Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2022

Mathematics: analysis and approaches

Higher level

Paper 2

32 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates

fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first"

Section A

1. (a) $1.01206\ldots, 2.45230\ldots$

$a = 1.01$, $b = 2.45$ $(1.01x + 2.45)$ ***A1A1***

[2 marks]

(b) $0.981464\ldots$

$r = 0.981$ ***A1***

Note: A common error is to enter the data incorrectly into the GDC, and obtain the answers $a = 1.01700\ldots, b = 2.09814\ldots$ and $r = 0.980888\ldots$ Some candidates may write the 3 sf answers, ie. $a = 1.02, b = 2.10$ and $r = 0.981$ or 2 sf answers, ie. $a = 1.0, b = 2.1$ and $r = 0.98$. In these cases award ***A0A0*** for part (a) and ***A0*** for part (b). Even though some values round to an accepted answer, they come from incorrect working.

[1 mark]

(c) correct substitution of 78 into **their** regression equation ***(M1)***

$81.3930\ldots$, 81.23 from 3 sf answer

81 ***A1***

[2 marks]

Total [5 marks]

2. (a) $\left(0.708519...,0.639580...\right)$

$\left(0.709,0.640\right)$ ($x=0.709$, $y=0.640$) ***A1A1***

[2 marks]

(b) $1.09885...$

$x=1.10$ (accept $\left(1.10,0\right)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 \left|f\left(x\right)\right| dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^{2} f\left(x\right) dx$ OR $\int_{1.09885...}^{2} \left|f\left(x\right)\right| dx$ OR 4.17527… ***(A1)***

$\int_0^{1.09885...} f\left(x\right) dx - \int_{1.09885...}^{2} f\left(x\right) dx$ OR $0.435901\ldots + 4.17527\ldots$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

Total [6 marks]

3. $86.4 = 50r^3$ ***(A1)***

$r = 1.2\left(= \sqrt[3]{\frac{86.4}{50}}\right)$ seen anywhere ***(A1)***

$\frac{50\left(1.2^n - 1\right)}{0.2} > 33500$ OR $250\left(1.2^n - 1\right) = 33500$ ***(A1)***

attempt to solve their geometric S_n inequality or equation ***(M1)***

sketch OR $n > 26.9045,\ n = 26.9$ OR $S_{26} = 28368.8$ OR $S_{27} = 34092.6$ OR algebraic manipulation involving logarithms

$n = 27$ (accept $n \geq 27$) ***A1***

Total [5 marks]

4. recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0) = 15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000 \times 0.89 (= 13350)$ ***(A1)***

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000 \times 0.89 = 15000\text{e}^{8k}$ OR $13350 = 15000\text{e}^{8k}$

$k = \dfrac{\ln 0.89}{8}\ (-0.014566)$ ***(A1)***

substitution of $t = 2041 - 2014 (= 27)$ and their value for k into the model ***(M1)***

$P(27) = 15000e^{-0.0145... \times 27}$

$10122.3\ldots$

$P(27) = 10100\ (10122)$ ***A1***

Total [7 marks]

5.

Note: Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n = 9$ and a power of ax) ***(M1)***

${}^9C_r (ax)^{9-r} (1)^r$ OR ${}^9C_{9-r} (ax)^r (1)^{9-r}$ OR ${}^9C_0 (ax)^0 (1)^9 + {}^9C_1 (ax)^1 (1)^8 + \dots$

recognizing that the term in x^6 is needed ***(M1)***

$\frac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r = 6$ OR $r = 3$ OR $9 - r = 6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6 (ax)^6 (1)^3$ OR ${}^9C_3\, a^6 x^6$ OR $84\left(a^6 x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\frac{{}^9C_6}{21} a^6 x^4$ OR $4a^6 x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\frac{8}{7} a^5 x^4$ or $\frac{8}{7} a^5$ (do not accept other powers of *x*) ***(M1)***

$\frac{{}^9C_3}{21} a^6 x^4 = \frac{8}{7} a^5 x^4$ OR $4a^6 = \frac{8}{7} a^5$

continued…

Question 5 continued

OR

correct term in x^6 or coefficient of x^6 (may be seen in equation) ***(A1)***

$84a^6x^6$ OR $84a^6$

Set their term in x^6 or coefficient of x^6 equal to $24a^5x^6$ or $24a^5$
(do not accept other powers of *x*) ***(M1)***

$84a^6x^6 = 24a^5x^6$ OR $84a = 24$

THEN

$a = \frac{2}{7} \approx 0.286(0.285714...)$ ***A1***

Note: Award ***A0*** for the final mark for $a = \frac{2}{7}$ and $a = 0$.

Total [6 marks]

6. (a) $\int_0^b axe^x \mathrm{d}x = 1$ (seen anywhere) ***M1***

attempt to use integration by parts (either way around) (***M1***)

$\left[axe^x\right]_0^b - \int_0^b ae^x \mathrm{d}x \left(= 1\right)$ ***(A1)***

$\left[axe^x\right]_0^b - \left[ae^x\right]_0^b \left(= 1\right)$ ***A1***

Note: Condone incorrect or absent limits up to this point.

$abe^b - ae^b + a = 1$

$a = \dfrac{1}{be^b - e^b + 1}$ ***A1***

[5 marks]

(b) $\int_0^m xe^x \mathrm{d}x = \dfrac{1}{2}$ ***(M1)***

$\left[xe^x\right]_0^m - \left[e^x\right]_0^m = \dfrac{1}{2}$

$me^m - e^m + 1 = \dfrac{1}{2}$ ***(A1)***

$m = 0.768039...$

$m = 0.768$ ***A1***

[3 marks]

Total [8 marks]

7. (a) **METHOD 1**

attempt to use scalar product or formula for angle between two vectors ***(M1)***

$\boldsymbol{u}\cdot\boldsymbol{v} = \cos\frac{1}{n} + \sin\frac{1}{n}$ (seen anywhere) ***(A1)***

$$\cos\theta = \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}\sqrt{\left(\cos^2\frac{1}{n} + \sin^2\frac{1}{n}\right)}} \left(= \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}}\right)$$ ***A1***

METHOD 2

attempt to use an Argand diagram showing two complex numbers in the first quadrant with the angle between them marked as θ ***(M1)***

$\arg(\boldsymbol{u}) = \frac{\pi}{4}$ (accept 45° or $\arctan(1)$) **and** $\arg(\boldsymbol{v}) = \frac{1}{n}$ ***(A1)***

$\cos\theta = \cos\left|\frac{\pi}{4} - \frac{1}{n}\right|$ ***A1***

continued…

Question 7 continued

(b) use of $\frac{1}{n} \to 0$ as $n \to \infty$ ***(M1)***

EITHER

$\left(\cos\theta \to\right)\frac{1}{\sqrt{2}}$ ***(A1)***

OR

$\left(\boldsymbol{v} \to\right)\boldsymbol{i}$ ***(A1)***

THEN

the limit is $\frac{\pi}{4}$ ***A1***

Note: Accept $45°$. Do not accept rounded values such as 0.785.

[3 marks]

Total [6 marks]

8. **EITHER**

$\left(\frac{\mathrm{d}V}{\mathrm{d}h}=\right)10\pi h-\pi h^2$ ***(A1)***

Note: This ***A1*** may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h}=76.5616...$.

attempt to use chain rule to find a relationship between $\frac{\mathrm{d}h}{\mathrm{d}t}$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}\left(=\frac{1}{\left(\frac{\mathrm{d}V}{\mathrm{d}h}\right)}\times\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

OR

attempt to differentiate $V=5\pi h^2-\frac{1}{3}\pi h^3$ throughout with respect to t ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t}=10\pi h\frac{\mathrm{d}h}{\mathrm{d}t}-\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

continued…

Question 8 continued

THEN

$\left(10\pi h-\pi h^{2}\right)\frac{\mathrm{d}h}{\mathrm{d}t}=2$ OR $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{2}{10\pi h-\pi h^{2}}$ ***(A1)***

Note: Award this ***A1*** if the correct expression is seen with their h already substituted.

attempt to solve $200=5\pi h^{2}-\frac{1}{3}\pi h^{3}$ ***(M1)***

$h=4.20648...$ ***(A1)***

Note: This ***(M1)(A1)*** can be awarded independently of all previous marks, and may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h}=76.5616...$ Ignore extra values of h -3.24 and 14.0.

$\frac{\mathrm{d}h}{\mathrm{d}t}=0.0261227...$

$\frac{\mathrm{d}h}{\mathrm{d}t}=0.0261\left(\mathrm{cms}^{-1}\right)$ ***A1***

[6 marks]

9. (a) (i) attempt to use the cosine rule ***(M1)***

$$\text{AC} = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\alpha}\left(= \sqrt{20 - 16\cos\alpha} = 2\sqrt{5 - 4\cos\alpha}\right)$$ ***A1***

(ii) $$\text{AC} = \sqrt{6^2 + 8^2 - 2(6)(8)\cos\beta}\left(= \sqrt{100 - 96\cos\beta} = 2\sqrt{25 - 24\cos\beta}\right)$$ ***A1***

(iii) $5 - 4\cos\alpha = 25 - 24\cos\beta$

$\alpha = \arccos\left(6\cos\beta - 5\right)$ ***A1***

[4 marks]

(b) attempt to find the sum of two triangle areas using $A = \frac{1}{2}ab\sin C$ ***(M1)***

Note: Do not award this ***M1*** if the triangle is assumed to be right angled.

$$\text{Area} = \frac{1}{2}(8)\sin\alpha + \frac{1}{2}(48)\sin\beta$$ ***(A1)***

attempt to express the area in terms of one variable only ***(M1)***

$= 4\sqrt{1 - \left(6\cos\beta - 5\right)^2} + 24\sin\beta$ or $4\sin\left(\arccos\left(6\cos\beta - 5\right)\right) + 24\sin\beta$ OR

$4\sin\alpha + 24\sqrt{1 - \left(\frac{5 + \cos\alpha}{6}\right)^2}$ or $4\sin\alpha + 24\sin\left(\arccos\left(\frac{5 + \cos\alpha}{6}\right)\right)$

Max area = $19.5959\ldots$

$= 19.6$ ***A1***

[4 marks]

Total [8 marks]

Section B

10.

> **Note**: Do not penalize for inclusion or non-inclusion of endpoints for probabilities using a normal distribution. For example, for $\mathrm{P}(T < 55 \mid T > 40)$ accept $\mathrm{P}(T \le 55 \mid T > 40)$, $\mathrm{P}(T \le 55 \mid T \ge 40)$, etc.

(a) recognising to find $\mathrm{P}(T > 40)$ ***(M1)***

$\mathrm{P}(T > 40) = 0.574136...$

$\mathrm{P}(T > 40) = 0.574$ ***A1***

[2 marks]

(b) attempt to multiply four independent probabilities using their $\mathrm{P}(T > 40)$ and $\mathrm{P}(T < 40)$ ***(M1)***

$(1-p)^3 \cdot p$ OR $(1-0.574136...)^3 \cdot 0.574136...$ OR $(0.425863...)^3 \cdot 0.574136...$ ***(A1)***

$0.0443430...$

0.0443 , 0.0444 from 3 sf values ***A1***

[3 marks]

continued…

Question 10 continued

(c) (i) recognizing conditional probability ***(M1)***

$\mathrm{P}(T < 55 \mid T > 40)$

Note: Award ***(M1)*** for an expression or description in context. Accept $\mathrm{P}(T > 40 \mid T < 55)$ but do not accept just $\mathrm{P}(A \mid B)$.

$\dfrac{\mathrm{P}(40 < T < 55)}{\mathrm{P}(T > 40)}$ ***(A1)***

$\dfrac{0.461944...}{0.574136...}$ ***(A1)***

$\mathrm{P}(T < 55 \mid T > 40) = 0.804590...$

$= 0.805$ ***A1***

(ii) recognizing binomial probability ***(M1)***

$X \sim \mathrm{B}(n,\ p)$

$n = 10$ and $p = 0.804589...$ ***(A1)***

$0.0242111...,0.0240188...\text{using } p = 0.805$

$\mathrm{P}(X = 5) = 0.0242$ ***A1***

[7 marks]

continued…

Question 10 continued

(d) Let $\mathrm{P}(T < a) = x$

recognition that probabilities sum to 1 (seen anywhere) ***(M1)***

EITHER

expressing the three regions in one variable ***(M1)***

$x + 0.904 + 2x$ OR $\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a)$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b)$

OR x and $2x$ correctly indicated on labelled bell diagram

$\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a) = 1$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b) = 1$ (or equivalent) ***(A1)***

OR

expressing either $\mathrm{P}(T < a)$ or $\mathrm{P}(T > b)$ only in terms of $\mathrm{P}(a \le T \le b)$ ***(M1)***

$\left(\mathrm{P}(T < a) =\right)\frac{1}{3}\left(1 - \mathrm{P}(a \le T \le b)\right)$ OR $\left(\mathrm{P}(T > b) =\right)\frac{2}{3}\cdot\left(1 - \mathrm{P}(a \le T \le b)\right)$

$x = \frac{1}{3}(1 - 0.904)(= 0.032)$ OR $\mathrm{P}(T > b) = \frac{2}{3}(1 - 0.904)(= 0.064)$ ***(A1)***

THEN

$\mathrm{P}(T < a) = 0.032$

$a = 22.18167...$

$a = 22.2$ (accept 22.1) ***A1***

[4 marks]

Total [16 marks]

11. (a) attempt to use product rule *(M1)*

$f'(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ *(M1)*

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ *(A1)*

attempt to use formula for volume of revolution *(M1)*

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$\pi\int_0^{\frac{4}{3}}\left(e^{2x}(3x-4)\right)^2 \mathrm{d}x$ *(A1)*

Note: This ***(A1)*** can be awarded if the $\mathrm{d}x$ is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

continued…

Question 11 continued

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 \left(= -\mathrm{e}^2\right)$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1)(= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 \left(= 2\mathrm{e}^2\right)$ ***A1***

[5 marks]

Total [15 marks]

12.

(a) $\overrightarrow{AB} = \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ ***A1A1***

[2 marks]

(b) **METHOD 1**

$k-1 = 2 \times 4$ ***M1***

$k = 9$ ***AG***

METHOD 2

in order by y or z-ordinate, the points are $(k, -2, 1), (5, 0, 2)\ (1, 2, 3)$

$k-5 = 5-1$ ***M1***

$k = 9$ ***AG***

[1 mark]

continued…

Question 12 continued

(c) (i) attempt to set up a vector equation using a point, a parameter and a direction vector ***(M1)***

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \text{ (or equivalent)}$$ ***A1***

Note: "$\boldsymbol{r} =$" or equivalent must be seen for ***A1***.

continued…

Question 12 continued

(ii) **METHOD 1**

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use a different parameter for L_2 ***(M1)***

$\frac{x-1}{2} = \frac{y}{3} = 1-z = \mu$ or $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

point on line L_2 has coordinates $(1+2\mu, 3\mu, 1-\mu)$ ***(A1)***

Note: This ***A1*** may be implied by $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

$1+4\lambda = 1+2\mu$

$2-2\lambda = 3\mu$

$3-\lambda = 1-\mu$

any two of the above equations ***A1***

attempt to solve two simultaneous equations with two parameters ***(M1)***

eg $\lambda = 0.25$, $\mu = 0.5$ or $\lambda = 1.6$, $\mu = -0.4$ or $\lambda = -2$, $\mu = -4$ ***A1***

substitute into third equation or solve a different pair of simultaneous equations ***M1***

obtain contradiction *eg* $3-0.25 \neq 1-0.5$ or $1+4(1.6) \neq 1+2(-0.4)$ or $2-2(-2) \neq 3(-4)$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

continued…

Question 12 continued

METHOD 2

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use the equation of L_2 to generate at least two equations in λ ***(M1)***

if the two lines intersect,

$$\frac{(1+4\lambda)-1}{2}=\frac{2-2\lambda}{3}\left(\Rightarrow 2\lambda=\frac{2-2\lambda}{3}\right)$$

$$\frac{(1+4\lambda)-1}{2}=1-(3-\lambda)(\Rightarrow 2\lambda=\lambda-2)$$

$$\frac{2-2\lambda}{3}=1-(3-\lambda)\Rightarrow\left(\frac{2-2\lambda}{3}=\lambda-2\right)$$

any two of the above equations ***A1A1***

attempt to solve at least one equation in λ ***(M1)***

one of $\lambda=\frac{1}{4}$, $\lambda=-2$, $\lambda=\frac{8}{5}$ seen ***A1***

substitute into second equation or solve second equation ***M1***

obtain contradiction *eg* $\lambda=\frac{1}{4}\neq-2$ or $2\left(\frac{1}{4}\right)\neq\frac{1}{4}-2$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

continued…

Question 12 continued

METHOD 3

attempt to use a find Cartesian equation for L_1 ***(M1)***

$$\frac{x-1}{4}=\frac{y-2}{-2}=\frac{z-3}{-1}$$ ***A1***

attempt to isolate one variable in both equations ***(M1)***

$L_1 : z=\frac{1-x}{4}+3=\frac{y-2}{2}+3 \quad L_2 : z=\frac{1-x}{2}+1=\frac{-y}{3}+1$ OR

$L_1 : y=\frac{1-x}{2}+2=2(z-3)+2 \quad L_2 : y=\frac{3(x-1)}{2}=3(1-z)$ OR

$L_1 : x=1-2(y-2)=1-4(z-3) \quad L_2 : x=\frac{2y}{3}+1=1-2(z-1)$ ***A1***

attempt to solve for each of the other two variables ***(M1)***

e.g. $\frac{1-x}{2}+1=\frac{1-x}{4}+3$ and $\frac{-y}{3}+1=\frac{y-2}{2}+3$

$x=-7$, $y=-1.2$ OR $x=2$, $z=1.4$ OR $y=1.5$, $z=5$ ***A1***

obtain contradiction *eg* $z=5\neq 1.4$ OR $y=1.5\neq -1.2$ OR $x=2\neq -7$

(so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

[10 marks]

continued…

Question 12 continued

(d) (i) **METHOD 1**

attempt to find cross product of two of $\vec{\mathrm{AB}}$, $\vec{\mathrm{AC}}$ and $\vec{\mathrm{BC}}$ or their opposites ***M1***

eg $\vec{\mathrm{AB}} \times \vec{\mathrm{AC}} = \begin{pmatrix} 0 \\ k-9 \\ 18-2k \end{pmatrix} \left(= (k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$ ***A1***

attempt to substitute their cross product and a point into the equation of a plane ***(M1)***

$(k-9)y + 2(9-k)z = 2(k-9) + 6(9-k)$

$(k-9)y + 2(9-k)z = 36-4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

METHOD 2

attempt to find vector equation of Π and write x, y and z in parametric form ***M1***

$\left(\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \Rightarrow \right) x = 1 + \lambda(k-1) + 4\mu,\ y = 2 - 4\lambda - 2\mu,$

$z = 3 - 2\lambda - \mu$ or equivalent ***A1***

attempt to eliminate both parameters to work towards Cartesian form ***M1***

$(k-9)y + 2(9-k)z = 36-4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to find the equation of the line through (0, 0, 0) perpendicular to the plane ***(M1)***

EITHER

$$(\boldsymbol{r} =)t\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$$t + 4t + 4 = 0$$

$$t = -\frac{4}{5}$$ ***(A1)***

OR

$$(\boldsymbol{r} =)t(k-9)\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$$t(k-9)^2 + 4t(k-9)^2 + 4(k-9) = 0$$

$$t = -\frac{4}{5(k-9)}$$ ***(A1)***

THEN

so the point on the plane closest to the origin is $(0\,,\ -0.8\,,\ 1.6)$ ***A1***

continued…

Question 12 continued

METHOD 2

choose a point on the plane (p , q , r)

$q-2r+4=0$ OR $q(k-9)-2r(k-9)+4(k-9)=0 \Rightarrow q=2r-4$

distance to the origin is $\sqrt{p^2+(2r-4)^2+r^2}$ ***(A1)***

since *p* is independent of *r*, distance is minimised when $p=0$ ***(R1)***

attempt to find the value of *r* for which their $\sqrt{(2r-4)^2+r^2}$ is minimised ***(M1)***

$r=1.6$ ***(A1)***

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

continued…

Question 12 continued

METHOD 3

attempt to find a vector from the origin to the closest point on the plane ***(M1)***

EITHER

$$(\boldsymbol{r} =) t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$ ***(A1)***

distance to the origin $= \left(\dfrac{4}{\sqrt{1^2 + (-2)^2}} = \dfrac{4}{\sqrt{5}}\right) = \dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t = \pm\dfrac{4}{5}$

check in equation of plane $y - 2z = -4$ to get $t = -\dfrac{4}{5}$ ***(R1)***

OR

$$(\boldsymbol{r} =) t(k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$ ***(A1)***

distance to the origin $= \left(\dfrac{4}{\sqrt{1^2 + (-2)^2}} = \dfrac{4}{\sqrt{5}}\right) = \dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t = \pm\dfrac{4}{5(k-9)}$

check in equation of plane $y - 2z = -4$ to get $t = -\dfrac{4}{5(k-9)}$ ***(R1)***

THEN

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

[9 marks]

Total [22 marks]